MATH 1010E University Mathematics
Lecture Notes (week 5)
Martin Li

1 Implicit Differentiation

Sometimes a function is defined implicitly by an equation of the form

$$f(x, y) = 0,$$

which we think of as a relationship between x and y, and we can "solve" y in terms of x to get a function $y = y(x)$ such that

$$f(x, y(x)) = 0 \qquad \text{for all } x.$$

This is how we understand that the equation $f(x, y) = 0$ "defines" the function $y(x)$ implicitly. Let us start with a simple example.

Example 1.1 *Consider the linear equation $4x + 3y = 5$, we can solve y in terms of x to get*

$$y = \frac{1}{3}(5 - 4x).$$

Hence, the equation defines a function $y(x) = \frac{1}{3}(5 - 4x)$, whose derivative is $y'(x) = -4/3$.

Question: Can we find the derivative $y'(x)$ without first solving y in terms of x from the equation $f(x, y) = 0$?

The answer is YES! And this can be done by "*implicit differentiation*". The idea is just to differentiation the whole equation $f(x, y) = 0$ with respect to x, keeping in mind that $y = y(x)$ is a function of x. For example, in Example 1.1, we have some implicitly defined function $y(x)$ such that

$$4x + 3y(x) = 5 \qquad \text{for all } x.$$

Differentiating both sides with respect to x, we get

$$4 + 3y'(x) = 0,$$

which implies $y'(x) = -4/3$. Note that we get the same answer without having to solve for y.

However, in some cases the equation $f(x, y) = 0$ may not define a unique function $y(x)$. Let's look at the example below.

Example 1.2 *Consider the equation*

$$x^2 + y^2 = 1,$$

then when we solve for y in terms of x, we get

$$y = \pm\sqrt{1 - x^2}.$$

Hence the equation defines implicitly $y_\pm(x) = \pm\sqrt{1-x^2}$ which both satisfies the equation $x^2 + y_\pm^2(x)) = 1$. Geometrically, y_+ and y_- defines the upper and lower unit circle respectively. If we calculate their derivatives, we get

$$y'_+(x) = -\frac{x}{\sqrt{1-x^2}},$$
$$y'_-(x) = \frac{x}{\sqrt{1-x^2}}.$$

If we do implicit differentiation instead, we differentiate the equation $x^2 + y^2 = 1$ with respect to x, applying chain rule when the derivative falls on y^2, we obtain

$$2x + 2yy' = 0.$$

We can solve y' in terms of x and y to get

$$y' = -\frac{x}{y}.$$

Note that both $y'_\pm$ satisfy the equation above.

Remark 1.3 *In general, implicit differentiation can only find y' in terms of both x and y. If we want to find y' solely in terms of x, we still have to solve y in terms of x, or in some case, we can simply the expression of y and x to an expression only involving x.*

Example 1.4 *Consider the equation*

$$y \sin y + x = 1,$$

it is impossible to solve for y in terms of x by elementary means. Therefore, we can only do implicit differentiation to find y'. Differentiating the equation, we get

$$y' \sin y + yy' \cos y + 1 = 0,$$

which we can solve for y' to obtain

$$y' = -\frac{1}{\sin y + y \cos y}.$$

Question: Why can we always solve for y' in terms of x and y after differentiating an equation $f(x, y) = 0$? (Hint: Think about how y' comes out as a consequence of chain rule.)

2 Higher Derivatives and Differential Equations

Recall that we can think of differentiation as a way to obtain a new function $f'(x)$ from an old one $f(x)$. From this point of view, we can keep on differentiating the functions further to obtain the *higher derivatives* of f. For example,

$$f(x) = x^2, \qquad f'(x) = 2x, \qquad f''(x) = 2.$$

Here, $f''(x)$ is the second derivative of f and we denote the n-th derivative of f by $f^{(n)}(x)$.

Remark 2.1 *Not all functions can be differentiated infinite number of times (counterexample?). When it can be differentiated indefinitely, we say that the function f is* smooth.

Example 2.2 *Consider the function $f(x) = \sin x$, differentiation gives*

$$f'(x) = \cos x \qquad \text{and } f''(x) = -\sin x.$$

Note that $f''(x) = -f(x) = -\sin x$. Therefore, f and f'' are related by the equation

$$f''(x) + f(x) = 0.$$

This is an example of a second order linear differential equation.

Question: Check that $f(x) = \cos x$ satisfies the same differential equation above. Does it exist other function satisfying the same differential equation? Can we write down all of them?

Question: What function satisfies the differential equation $f'(x) - f(x) = 0$? Can you write down all such functions? What about $f'(x) - 2f(x) = 0$?

Remark 2.3 *A function $f(x)$ "satisfies" a differential equation means that when we substitute $f(x)$ and its derivatives into the equation, we obtain an identity in x, not just an equation that only holds for some particular values of x.*

3 Optimization I - First Order Condition

One major application of differentiation is that it can help us find the maximum and minimum of a function. This is particularly useful in daily life, e.g. we often want to minimize the cost $f(x)$ in terms of the parameter x in business. The typical question we are looking at is

$$\text{min/max} \qquad f(x) \qquad \text{on some interval } I.$$

In practice, the interval I could be a finite or infinite interval. Depending on the situation, a minimum or maximum may or may not exist. If it exists, it may lie in the interior of I or at the boundary. Roughly speaking, the technique of differentiation helps us locate minimum/maximum that lies in the interior of I. Boundary points of I have to be treated separately.

Definition 3.1 *We say that $x_0 \in I$ is a minimum (resp. maximum) point of $f(x)$ in I if $f(x) \geq f(x_0)$ (resp. $f(x) \leq f(x_0)$) for all $x \in I$. A minimum or maximum point is called an extremum point.*

Theorem 3.2 (First order condition) *Suppose f in differentiable in the interior of I and x_0 is an extremum point lying in the interior of I, then $f'(x_0) = 0$. We say that x_0 is a critical point of f if $f'(x_0) = 0$.*

In other words, any interior extremum point is a critical point of f. Note that not all critical points of f are extremum points, they are just possible candidates for extremum points.

Question: Give an example of a function f which has a non-extremum but critical point.

The idea is that to find the maximum and minimum of f, we just have to locate all interior critical points of f. Together with the boundary points, we compare to see which one has the largest and smallest value of f. This would give the minimum and maximum. Let us look at an example.

Example 3.3 *Find the maximum and minimum of $f(x) = x^2 + x + 2$ on the interval $[-1, 1]$.*

Solution: First, we locate all the interior critical points of f by solving

$$f'(x) = 2x + 1 = 0.$$

The only solution is $x = -1/2$ hence we have only one interior critical point at $x = -1/2$. There are two boundary points $x = \pm 1$. If we compare the values at these points:

$$f(-1/2) = 7/4, \qquad f(-1) = 2, \qquad f(1) = 4.$$

We see that the maximum is 4 which is achieved at the boundary point $x = 1$, and the minimum is $7/4$ which is achieved at the interior critical point $x = -1/2$.

If the interval I is infinite, then the minimum or maximum may not exist. For example, the function $f(x) = x$ on $I = (-\infty, \infty)$ does not have any minimum or maximum points. However, sometimes we can still find a min/max if we know the limit behavior of the function near infinity.

Example 3.4 *Find the maximum and minimum of the function* $f(x) = x^2 e^{-x}$ *on the interval* $I = [-1, \infty)$.

Solution: First we locate all the interior critical points by solving

$$f'(x) = 2xe^{-x} - x^2 e^{-x} = 0.$$

Since $e^{-x} > 0$ for any x, we can cancel it out to get $2x - x^2 = 0$, which gives two critical point $x = 0$ and $x = 2$. There is only one finite boundary point $x = -1$. Let us first compute the values of f at these three points:

$$f(0) = 0, \qquad f(2) = 4e^{-2}, \qquad f(-1) = e.$$

For the behavior of f near $+\infty$, we look at the limit

$$\lim_{x \to +\infty} x^2 e^{-x} = 0.$$

The limit is zero since as $x \in +\infty$, e^x goes to $+\infty$ faster than ANY polynomial. (This limit can be evaluated using L'Hospital's rule later.) We also observe that

$$f(x) = x^2 e^{-x} \geq 0 \qquad \text{for all } x \in I.$$

Therefore, combining all the discussions above, we have the maximum is e located at the boundary point $x = -1$ and the minimum is 0 located at the interior critical point $x = 0$.